The geometry of gauge - particle field interaction: a generalization of Utiyama's theorem

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Abstract. *The paper classifies the locally gauge invariant Lagrangians on the jet bundle $J^1(E \oplus C)$, for interacting particle and gauge fields. This serves to clarify the global nature of the Utiyama extension process (Yang-Mills trick) for arbitrary principal bundles P and gives the classical (local) results when P is trivial: $P = M \times G$. The emphasis of the paper is a formulation of the results in terms of geometric objects on associated bundles over M rather than on bundles over P.*

I. INTRODUCTION

In his original paper [2] on invariant interactions, Utiyama proved a theorem (now known as Utiyama's theorem) that classified those Lagrangians for the gauge fields which are locally gauge invariant. While Utiyama's arguments are local in nature (relying on a trivial bundle structure) global formulations of this theorem and other aspects of his paper for non-trivial bundles subsequently have appeared in numerous places in the literature (Cf. Bleecker [2], Garcia [3, 4, 5], Garcia and Perez-Rendon [6], Parker [7] and Mangiarotti and Modugno [8], and Modugno [9], for example).

In this paper, we prove the following generalization of Utiyama's (classification) theorem: A Lagrangian L^+ for the particle fields τ interacting with gauge

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fields σ is locally gauge invariant if and only if it is a function of only the particle fields τ, their covariant derivatives $\nabla^{\sigma}\tau$, and the curvature F^{σ} of the gauge fields σ. We prove this at the group level rather than by passing to vector fields (infinitesimal gauge transformations), illustrating the prominence of the gauge orbit structure in the result and avoiding certain intricacies with partial differential equations.

The paper also shows how to generalize the Utiyama extension process (Yang-Mills trick) for obtaining a locally gauge invariant Lagrangian L_1^{γ} for particle-gauge interaction from a «globally» gauge invariant Lagrangian L_1 for the «free» particles. Since for non-trivial bundles, global gauge transformations are non-existent, we show, using the classification theorem, that an appropriate generalization involves invariance of L_1 under a certain set G_{γ} of local bundle maps. Our approach here involves a fixed background gauge field (connection) γ, which seems indigenous to the global theory (γ is the trivial connection in Utiyama's case). The classification theorem also serves to shed some light on the topic of «minimal» interaction Lagrangians for the particle and gauge fields.

An additional purpose of ther paper is to emphasize an appropriate fiber bundle formalism for expressing the global aspects of the gauge theory (no local coordinates, trivial bundle structure, etc.). Our approach is essentially the approach originated by Garcia [3] (also cf. Modugno [9], Mangiarotti and Modugno [8]). The particle and gauge fields are sections $\tau : M \to E$, $\sigma : M \to C$ of certain fiber bundles E, C over the base manifold M (spacetime). Then an interaction Lagrangian is a smooth map $L : J^1(E \oplus C) \to \mathbb{R}$ on the first order jet bundle of their fibered product: $E \oplus C$. The other prominent approach to a global formulation of the theory is to view the gauge fields as certain 1-forms on a principal bundle P over M (cf. Bleecker [2]), and in this approach the appropriate bundles are equivariant bundles over P. The connection of this with the former approach is briefly delineated in Sections III and IV below. Basically the natural equivalence between the bundles in the equivariant category and their associated bundles over M allows one to formulate the theory in either category.

After submission of this paper it was brought to our attention that the recent work by Mangiarotti [18, 19] and Modugno [20] extends their previous work on free gauge fields (Ref.'s 8-9) to gauge fields interacting with particle fields. Their work will be briefly described in the conclusion (Section VIII) for the sake of comparison.

II. A GEOMETRICAL STRUCTURE FOR THE THEORY

In this section we present the main body of our results and the indication of their proofs, relegating the precise definitions, descriptions of notation, and further details to the ensuing sections.

The natural tool for our formulation of the gauge theory is the functor from the category of equivariant bundles $S \to P$ over a fixed principal bundle $P = P(M, G) \xrightarrow{\pi} M$ into the category of bundles over M, wich takes S into its associated bundle $S/G \to M$. Thus for us the gauge fields are sections of the connection bundle $C = A^1_{HC}(P, TP)/G$ and the particle fields are sections τ of an appropriate vector bundle $E = (P \times F)/G$ (Cf. also Ref's 3, 6 for related developments on this approach). While one can formulate the theory in the equivariant category (as, for example, Bleecker [2] does) we find the associated bundles more convenient since then the fields are functions (sections) on spacetime M and one has available all the standard variational calculus (Euler-Lagrange equations, symmetries and conservation laws) based on the jet bundles J^1E, J^1C, etc. (for this Cf., for example, Hermann [10, 11], Garcia [12], Krupka [13], Betounes [14, 15, 16]). In the equivariant category one must develop the variational theory from scratch (Bleecker [2], Parker [7], etc.).

The global formulation of Utiyama's theorem involves the curvature map $\Omega : J^1C \to A^2(M, AdP)$ defined by

(2.1) $$\Omega([\sigma]_x = F^\sigma(x)$$

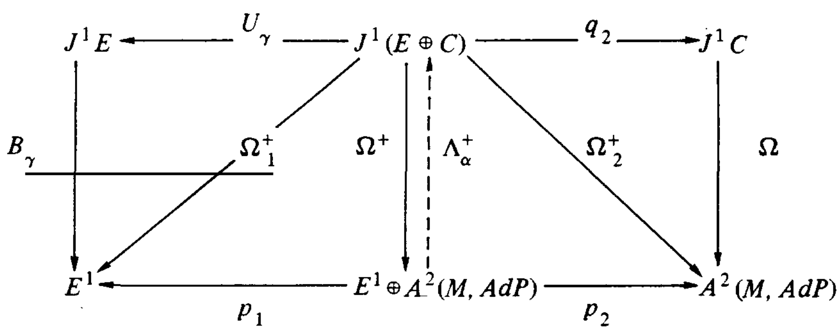
(Cf. Garcia [3], Garcia and Perez-Rendon [6], Mangiarotti and Modugno [8]). With $\oplus$ denoting the fibered product, Ω extends naturally to a bundle epimorphism $\Omega^+ : J^1(E \oplus C) \to E^1 \oplus A^2(M, AdP)$ (with $E^1 \equiv E \oplus A^1(M, E)$) defined by

(2.2) $$\Omega^+([\tau]_x, [\sigma]_x) = (\tau(x), \nabla^\sigma\tau(x), F^\sigma(x))$$

Furthermore, each fixed connection $\gamma : M \to C$ gives a natural bundle isomorphism $B_\gamma : J^1E \to E^1 = E \oplus A^1(M, E)$ defined by

(2.3) $$B_\gamma([\tau]_x) = (\tau(x), \nabla^\gamma\tau(x))$$

Thus one gets the following commutative diagram:

(2.4)


where p_1, p_2, q_2 are the natural projections, $\Omega_1^+ = p_1 \circ \Omega^+$, $\Omega_2^+ = p_2 \circ \Omega^+$, and $U_\gamma \equiv B_\gamma^{-1} \circ \Omega_1^+$ (the Utiyama extension map).

Now there are representations of the automorphism group Aut(P) of P (and thus its subgroup $GA(P)$ of gauge transformations of P) on the associated bundles: C, E, E^1, $A^2(M, AdP)$ and their fibered products. For simplicity we denote the representations of the $\rho \in \mathrm{Aut}(P)$ again by ρ, and the prolongations to the respective jet bundles by ρ^1. The Lagrangians of interest to us here are those which are invariant ($\mathscr{L}_Z L = 0$) under *infinitesimal* gauge transformations Z (a representation of a vertical right invariant vector field Z on P). However, since the flow generated by Z consists of only locally defined gauge transformations ρ_t, we concentrate on the notion of local gauge invariance: A *local gauge transformation* of P is a local diffeomorphism ρ of a restricted subbundle $P|\, W = \pi^{-1}(W)$ over some open $W \subset M$, such that $\rho(ug) = \rho(u)g$ and $\pi \circ \rho = \pi$. We denote the set of local gauge transformations by $GA(P)_{\mathrm{loc}}$. The above representations of Aut(P) on associated bundles carry over to $GA(P)_{\mathrm{loc}}$ as well, and a function K on such an associated bundle (or a Lagrangian L on its jet bundle) is called *locally gauge invariant* if $K \circ \rho = K (L \circ \rho^1 = L)$ for every $\rho \in GA(P)_{\mathrm{loc}}$.

The results of the paper can now be stated as follows

THEOREM 1. *The extended curvature map Ω^+ intertwines the respective representations:*

$$\Omega^+ \circ \rho^1 = \rho \circ \Omega^+$$

for every $\rho \in$ Aut(P) (and also for local automorphisms of P).

The proof of this is given in Section IV. As a corollary one can now easily prove

THEOREM 2. *If a Lagrangian $L^+ : J^1(E \oplus C) \to \mathbb{R}$ factors through Ω^+ :*

$$L^+ = K^+ \circ \Omega^+$$

for some $K^+ : E^1 \oplus A^2(M, AdP) \to \mathbb{R}$, then $L^+ \circ \rho^1 = L^+$ if and only if $K^+ \circ \rho = K^+$. In particular L^+ is locally gauge invariant if and only if K^+ is. ■

Our main result is

THEOREM 3. *Each locally gauge invariant Lagrangian $L^+ : J^1(E \oplus C) \to \mathbb{R}$ factors through Ω^+ :*

$$L^+ = K^+ \circ \Omega^+ .$$

By theorem 2, K^+ is necessarily locally gauge invariant.

We prove this theorem via:

LEMMA 1: *For each chart $U_\alpha \subset M$ there exists a local bundle map*

$$\Lambda^+_\alpha : E^1 \oplus A^2(M, AdP) \to J^1(E \oplus C)$$

(defined on the fiber over U_α) such that

(a) $\Omega^+ \circ \Lambda^+_\alpha = 1$

(b) *for each* $([\tau]_x, [\sigma]_x) \in J^1(E \oplus C), x \in U_\alpha$ *there exists a local gauge transformation ρ such that*

$$\Lambda^+_\alpha \circ \Omega^+([\tau]_x, [\sigma]_x) = \rho^1([\tau]_x, [\sigma]_x).$$ ■

With this Lemma one now easily has:

Proof of Theorem 3: on the fibers over U_α define K^+_α by $K^+_\alpha = L^+ \circ \Lambda^+_\alpha$. Then since L^+ is locally gauge invariant, part (b) of Lemma 1 shows that

$$K^+_\alpha \circ \Omega^+ = L^+ \circ \Lambda^+_\alpha \circ \Omega^+ = L^+$$

on the fibers over U_α. But since Ω^+ is an epimorphism, this last equation shows that $K^+_\alpha = K^+_\beta$ on the fibers over $U_\alpha \cap U_\beta$. Thus one gets a globally defined map K^+ such that $K^+ \circ \Omega^+ = L^+$.

Putting theorems 2 and 3 together then gives the classification theorem: A Lagrangian L^+ on $J^1(E \oplus C)$ is locally gauge invariant if and only if there exists a locally gauge invariant function K^+ on $E^1 \oplus A^2(M, AdP)$ such that $L^+ = K^+ \circ \Omega^+$, i.e.

$$L^+([\tau]_x, [\sigma]_x) = K^+(\tau(x), \nabla^\sigma \tau(x), F^\sigma(x)).$$ ■

As a corollary (by, say, taking $E = O$) one obtaines:

UTIYAMA'S THEOREM: A Lagrangian $L_2 : J^1 C \to \mathbb{R}$ is locally gauge invariant if and only if there exists a locally gauge invariant function $K_2 : A^2(M, AdP) \to \mathbb{R}$ such that

$$L_2 = K_2 \circ \Omega$$

i.e.

$$L_2([\sigma]_x) = K_2(F^\sigma(x)).$$ ■

Finally, the other side of the diagram (2.4) gives a prescription for the:

UTIYAMA EXTENSION PROCESS: with respect to a given connection $\gamma : M \to C$, each Lagrangian $L_1 : J^1 E \to \mathbb{R}$ for the particle fields extends to a Lagrangian $L_1^\gamma : J^1(E \oplus C) \to \mathbb{R}$ for interacting particle and gauge fields. This Lagrangian is defined by

$$L_1^\gamma = L_1 \circ U_\gamma$$

where

$$U_\gamma = B_\gamma^{-1} \circ \Omega_1^+ = B_\gamma^{-1} \circ p_1 \circ \Omega^+$$

is the Utiyama map (relative to γ).

Examination of U_γ in local coordinates (Cf. Section VI) shows that this is an appropriate generalization of Utiyama's construction and reduces to his case ($P = M \times G$) when γ is the trivial connection on $M \times G$. The presence of γ in the extension is perhaps disconcerting, but for non-trivial bundles this is in the nature of things (there is no canonical isomorphism $J^1E \to E^1$). If one wishes to change the rules, this difficulty can be obviated by just considering functions $K_1 : E^1 \to \mathbb{R}$ as representing the particle Lagrangians (Cf. Ref. 6). Utiyama's result that the extension L_1^γ is locally gauge invariant when L_1 is globally gauge invariant generalizes as follows. For non-trivial principal bundles the notion of a global gauge transformation makes no sense, and so we proceed to get the correct invariance concept for L_1 from the classification theorem. Noting that

$$L_1^\gamma = K_1^\gamma \circ p_1 \circ \Omega^+$$

where $K_1^\gamma = L_1 \circ B_\gamma^{-1}$, we see by Theorem 2 that L_1^γ is locally gauge invariant if and only if $K_1^\gamma \circ p_1$ is; and of course $K_1^\gamma \circ p_1$ is locally gauge invariant if and only if K_1^γ is. But.

$$\begin{aligned} K_1^\gamma \circ \rho &= L_1 \circ B_\gamma^{-1} \circ \rho \\ &= (L_1 \circ B_\gamma^{-1} \circ \rho \circ B_\gamma) \circ B_\gamma^{-1} \end{aligned}$$

and so $K_1^\gamma \circ \rho = K_1^\gamma$ if and only if $L_1 \circ B_\gamma^{-1} \circ \rho \circ B_\gamma = L_1$. Thus:

THEOREM 4: *The extension L_1^γ is locally gauge invariant if and only if L_1 is invariant under*

$$G_\gamma = \{G_\gamma^{-1} \circ \rho \circ B_\gamma \mid \rho \in GA(P)_{\text{loc}}\} \qquad \blacksquare$$

By examining the coordinate expression for the local bundle map $B_\gamma^{-1} \circ \rho \circ B_\gamma : J^1E \to J^1E$ (Section VI) one sees that invariance of L_1 under G_γ is an appropriate generalization of Utiyama's global gauge invariance, and reduces to his case ($P = M \times G$) when $\gamma =$ the trivial connection.

The final stage in Utiyama's extension process is to add on to L_1^γ a locally gauge invariant Lagrangian for the gauge fields, which by Utiyama's theorem has the form $K_2 \circ \Omega_2^+ = K_2 \circ p_2 \circ \Omega^+$. Thus one obtains a locally gauge invariant Lagrangian of the form

$$L^+ = (K_1^\gamma \circ p_1 + K_2 \circ p_2) \circ \Omega^+ ,$$

which by our classification theorem has, among all the locally gauge invariant Lagrangians on $J^1(E \oplus C)$, a «minimal» amount of interaction between the particle and gauge fields (we do not give a precise meaning to «minimal» interaction here).

III. EQUIVARIANT BUNDLES AND THEIR ASSOCIATED BUNDLES

The equivariant category is the underlying format for the differential geometric aspects of gauge theory, and we review here a few main features of this, some of which can be found in the standard texts by Bleecker [2] and Kobayashi and Nomizu [17].

For any fixed base space M, consider the category of fiber bundles $p : E \to M$ over M with the set of morphisms $B(E, \bar{E})$ consisting of the fiber preserving diffeomorphisms (bundle maps) $f : E \to \bar{E}$. The diffeomorphism induced by f on the base space is denoted by $f_M : M \to M$. Letting $\Gamma(E)$ denote the set of sections $\tau : M \to E$, one gets for each $f \in B(E, \bar{E})$ a map (pullback map) $f^* =$ $= \Gamma(\bar{E}) \to \Gamma(E)$ defined by

$$f^*(\tau) = f^{-1} \circ \tau \circ f_M$$

When E is a vector bundle, we let $A^k(M, E)$ denote the bundle over M with fibers $A^k(M, E)_x$ consisting of the k-linear, anti-symmetric maps $\theta_x : TM_x \times \ldots \times TM_x \to E_x$. Then each $f \in B(E, \bar{E})$ gives rise to a bundle map $A^k(f) : A^k(M, E) \to A^k(M, \bar{E})$ refined by $A^k(f)(x, \theta_x), = (f_M(x), f \circ \theta_x \circ df_M^{-1}|^k_{f_M(x)})$.

The sections of $A^k(M, E)$ will be denoted by $\Lambda^k(M, E)$ ($\Lambda^k = \Gamma \circ A^k$), and are the differential k-forms on M with values in $\Gamma(E)$. For $f \in B(E, E)$, the pullback map $A^k(f)^* : \Lambda^k(M, \bar{E}) \to \Lambda^k(M, E)$ is

$$A^k(f)^*(\theta) = A^k(f^{-1}) \circ \theta \circ f_M$$

Throughout the paper we suppose that G is a Lie group (with Lie algebra $\mathcal{G}$) and that $\pi : P \to M$ is a principal G-bundle over M with right G-action denoted by $R_g u = ug$. If S and Q are manifolds with right G-actions, then a map $h : S \to Q$ is called *equivariant* if $h(sg) = h(s)g$ for every $s \in S$, $g \in G$. In particular the automorphism group Aut(P) of P consists of all the equivariant diffeomorphisms $\rho : P \to P$. The subgroup $GA(P)$ of gauge transformations of P consists of those automorphisms which induce the identity on the base space M. An *equivariant bundle* over P is a fiber bundle $\delta : S \to P$ with right G-action such that δ is an equivariant map. The *equivariant cateogry* has the equivariant bundles over P as objects and the equivariant diffeomorphisms $B_e(S, \bar{S})$ as morphisms. For each S in theis category oen gets an *associated bundle* $\delta' : S/G \to M$, where S/G is the set of equivalence classes $[s]$ ($s \sim s'$ if $s' = sg$ for some $g \in G$), and $\delta'([s]) \equiv \pi \circ \delta(s)$. This association $S \to S/G$ gives a covariant functor which takes a morphism $f \in B_e(S, \bar{S})$ over into the bundle map $f_G : S/G \to \bar{S}/G$ defined by $f_G[s] = [f(s)]$. The basic result for associated bundles is that the sections of S/G are in 1-1 correspondence with the *equivariant sections* $\Gamma_e(S)$ of S:

PROPOSITION 1. *There is a bijection*

$$\mu : \Gamma_e(S) \to \Gamma(S/G)$$

such that

$$f_G^* \circ \mu = \mu \circ f^*$$

for every $f \in B_e(S, \bar{S})$. *Thus* μ *is a natural equivalence between the functors* Γ_e *and* $\Gamma \circ (\cdot/G)$. ∎

When S is an equivariant *vector* bundle the above considerations apply equally well to the equivariant bundle $A^k(P, S)$ whose right action is given by $A^k(\tilde{R}_g)$, where $\tilde{R}_g$ denotes the right action on S. This action restricts to the subbundle $A_H^k(P, S)$ consisting of $(u, \theta_u) \in A^k(P, S)_u$ where θ_u is *horizontal*: $\theta_u(Z_u^1, \ldots, Z_u^k) = 0$ whenever one of the Z_u^i's is *vertical*: $d\pi|_u Z_u^i = 0$ (i.e., $Z_u^i \in VP_u$ where VP is the vertical subbundle of TP). There is then a natural equivalence

$$\nu : A_H^k(P, S)/G \to A^k(M, S/G)$$

$$\nu \circ A^k(f)_G = A^k(f_G) \circ \nu$$

and so in the sequel we will identify these bundles. Let $\Lambda_{He}^k(P, S)$ denote the equivariant sections of $A_H^k(P, S)$ (the horizontal, equivariant, k-forms on P with values in $\Gamma(S)$). The analog of Proposition 1 is

PROPOSITION 2. *There is a bijection*

$$\mu^k : \Lambda_{He}^k(P, S) \to \Lambda^k(M, S/G)$$

such that

$$\mu^k \circ A^k(f)^* = A^k(f_G)^* \circ \mu^k$$

(so μ^k *is a natural equivalence).* ∎

IV. THE CONNECTION BUNDLE AND PROOF OF THEOREM 1

We now specialize the foregoing to the bundles of concern for this paper. For the sake of simplifying the notation (but at the risk of adding confusion), we will denote the natural equivalence μ^k in Proposition 2 simply by μ and the pullbacks like $A^k(f)^*$ by f^*. Furthermore the equivariant bundles S of concern here are such that each $\rho \in \mathrm{Aut}(P)$ gives a $\tilde{\rho} \in B_e(S)$ with $\tilde{\rho}_P = \rho$. The map $\tilde{\rho}$ together with the corresponding map $\tilde{\rho}_G : S/G \to S/G$ will both be denoted simply by ρ. With these conventions, the identity in Proposition 2 reads

$$\rho^* \circ \mu = \mu \circ \rho^*$$

The specific bundles we need are the following:

(1) Suppose F is a finite dimensional vector space with left linear G-action $B : G \times F \to F$, $B(g, \xi) = g\xi$. Then $P \times F$ is an equivariant bundle with right action: $(u, \xi)g \equiv (ug, g^{-1}\xi)$. We denote the corresponding associated bundle by $E = (P \times F)/G$. For $\rho \in \mathrm{Aut}(P)$, one gets an equivariant bundle map of $P \times F$: $\rho(u, \xi) = (\rho(u), \xi)$. Using the canonical identification of F with the fibers of $P \times F$, one has $A^k(P, P \times F) \simeq A^k(P, F)$ and so $\rho(u, \theta_u) = (\rho(u), \theta_u \circ d\rho^{-1}|_{\rho(u)})$. Then for an F-valued k-form $\theta \in \Lambda^k(P, F)$, $(\rho^*\theta)_u = \theta_{\rho(u)} \circ d\rho|_u^k$.

(2) The tangent bundle TP is an equivariant bundle with right action $(u, Z_u)g = (g, dR_g|_u Z_u)$. This action restricts to the vertical subbundle $VP \subset TP$. Each $\rho \in \mathrm{Aut}(P)$ extends to an equivariant bundle map of TP (and VP) given by $\rho(u, Z_u) = (\rho(u), d\rho|_u Z_u)$. By the foregoing, the sections of TP/G correspond to equivariant (i.e. right invariant) vector fields Z on $P : Z_{ug} = dR_g|_u Z_u$. (These are the infinitesimal automorphisms of P). The bundle VP/G is called the *adjoint bundle* (gauge algebra) and the sections of it correspond to vertical equivariant vector fields on P (the infinitesimal gauge transformations of P). We take for the Lie algebra $\mathcal{G}$ of G the set of left invariant vector fields on G. The map $\Gamma_u : G \to P$, $\Gamma_u(g) = ug$ gives a representation $\Gamma : \mathcal{G} \to \mathrm{Vect}(P)$ of $\mathcal{G}$ as fundamental vector fields on $P : \Gamma(\xi)_u \equiv d\Gamma_u|_e \xi_e$. Now $P \times \mathcal{G}$ is an equivariant bundle with right action: $(u, \xi)g = (ug, Ad_{g^{-1}}\xi)$, and there is an equivariant bundle isomorphism $Q : P \times \mathcal{G} \to VP$ given by $Q(u, \xi) = (u, \Gamma(\xi)_u)$. Consequently $AdP = VP/G \simeq (P \times \mathcal{G})/G$.

(3) From the foregoing generalities, the equivariant bundle $A^1(P, TP)$ has its right action given by $(u, b_u)g = (ug, dR_g|_u \circ b_u \circ dR_{g^{-1}}|_{ug})$ and the action of $\rho \in \mathrm{Aut}(P)$ on $A^1(P, TP)$ works out to be $\rho(u, b_u) = (\rho(u), d\rho|_u \circ b_u \circ d\rho^{-1}|_{\rho(u)})$. Consequently the pullback of sections $b : P \to A^1(P, TP) : b(u) = (u, b_u)$ is given by $\rho^*(b)_u = d\rho^{-1}|_{\rho(u)} \circ b_{\rho(u)} \circ d\rho|_u$. All of these actions

restrict to the equivariant subbundle: $A^1_{HC}(P, TP) = \{(u, h_u) \mid h_u$ is horizontal and $d\pi|_u \circ h_u = d\pi|_u\}$. Each h_u projects TP_u on a complementary subspace to VP_u and $v_u \equiv 1 - h_u : TP_u \to VP_u$ restricts to the identity on VP_u. The bundle $A^1_I(P, VP)$ of such elements (u, v_u) is then an equivariant bundle isomorphic to $A^1_{HC}(P, TP)$. We call $A^1_{HC}(P, TP)$ the *Ehresmann bundle* and the equivariant sections of it: $h : P \to A^1_{HC}(P, TP)$, *Ehresmann connections*. The corresponding associated bundle

$$C \equiv A^1_{HC}(P, TP)/G \subset A^1(M, TP/G)$$

is called the *connection bundle*, and its sections $\sigma : M \to C$, *connections on* M (they are certain 1-forms on M with values in $\Gamma(TP/G)$). Each Ehresmann connection h gives an equivariant section $v^h : P \to A^1_I(P, VP)$ $(v^h_u = 1 - h_u)$ and using the isomorphism $Q : P \times \mathscr{G} \to VP$, one gets the corresponding *connection form* ω^h. This is the equivariant section $\omega^h : P \to A^1(P, \mathscr{G}) \simeq A^1(P, P \times \mathscr{G})$ defined by $\omega^h(Z) = Q^{-1}(v^h(Z))$. All of these different ways of viewing connections: h, v^h, ω^h, $\{h_u(TP_u)\}_{u \in P}$ are of course equivalent, but we take Ehresmann connections as fundamental since they correspond to connections σ on $M : h = \mu^{-1}(\sigma)$.

For an Ehresmann connection h one gets the usual covariant derivative $D^h : \Lambda^k_e(P, F) \to \Lambda^{k+1}_{He}(P, F)$ defined by $D^h = H^h \circ d$ where H^h is the horizontalization operator on forms $H^h(\theta)(Z^1, \ldots, Z^m) = \theta(hZ^1, \ldots, hZ^m)$. Thus each connection $\sigma : M \to C$ gives a covariant derivative $\nabla^\sigma : \Lambda^k(M, E) \to \Lambda^{k+1}(M, E)$ defined by

$$\nabla^\sigma \tau = \mu[D^{\mu^{-1}(\sigma)} \mu^{-1}(\tau)].$$

In addition the curvature of h is the form $\Omega^h \equiv D^h \omega^h \in \Lambda^2_{He}(P, \mathscr{G})$ and the corresponding curvature of σ is

$$F^\sigma = \mu[\Omega^{\mu^{-1}(\sigma)}].$$

The transformation properties of these operators and forms under automorphisms $\rho \in \mathrm{Aut}(P)$ is as follows. First an easy calculation shows that

$$H^{\rho^* h} = \rho^* \circ H^h \circ \rho^{-1*}$$

and consequently since d commutes with pullbacks

$$D^{\rho^* h} = \rho^* \circ D^h \circ \rho^{-1*}.$$

Hence for $\theta \in \Lambda^k_e(P, F)$

$$D^{\rho^* h}(\rho^* \theta) = \rho^*(D^h \theta).$$

A short computation shows that the connection form ω^h transforms according to

$$\omega^{\rho^* h} = \rho^*(\omega^h)$$

and consequently

$$\Omega^{\rho^* h} = \rho^*(\Omega^h)$$

Passing now to the associated bundles and using the fact (Propositions 1 and 2) that μ is a natural equivalence, one finds that

$$\nabla^{\rho^*\sigma}(\rho^*\tau) = \rho^*(\nabla^\sigma \tau)$$

for $\tau \in \Lambda^k(M, E)$ and

$$F^{\rho^*\sigma} = \rho^*(F^\sigma)$$

With these identities the proof of Theorem 1 (Section II) is as follows.

First recall that for a fiber bundle $Q \to M$ over M, the *1st order jet bundle* J^1Q consists of equivalence classes $[\tau]_x$, $x \in M$ of local sections $\tau : M \to Q$. Two sections τ, $\bar{\tau}$ being equivalent at x if they, as well as their first order partial derivatives, have the same values at x. A bundle map $f : Q \to Q$ then prolongs to a bundle map $f^1 : J^1Q \to J^1Q$ defined by

$$f^1([\tau]_x) = [f^{-1*}\tau]_{f_M(x)}$$

Putting everything together. the proof of Theorem 1 is

$$\begin{aligned}\Omega^+ \circ \rho^1([\tau]_x, [\sigma]_x) &= \Omega^+([\rho^{-1*}\tau]_{\rho_M(x)}, [\rho^{-1*}\sigma]_{\rho_M(x)}) \\ &= ((\rho^{-1*}\tau)(\rho_M(x)), (\nabla^{\rho^{-1*}\sigma}\rho^{-1*}\tau)(\rho_M(x)),\ F^{\rho^{-1*}\sigma}(\rho_M(x))) \\ &= (\rho(\tau(x)), \rho(\nabla^\sigma\tau(x)), \rho(F^\sigma(x))) \\ &= \rho \circ \Omega^+([\tau]_x, [\sigma]_x)\end{aligned}$$

V. COORDINATE EXPRESSIONS

For some of the ensuing work we will need the following coordinate expressions. Let $\phi_\alpha : U_\alpha \times G \to P|\,U_\alpha \equiv \pi^{-1}(U_\alpha)$ be a local trivialization of P, $\phi_{\alpha x}(g) \equiv \phi_\alpha(x, g)$, so $\phi_{\alpha x} : G \to \pi^{-1}\{x\}$ is a diffeomorphism. Assume that each $U_\alpha \subset M$ is a chart on M with coordinate functions x_i, $i = 1, \ldots, \dim(M)$, and suppose that (V, y^a), $a = 1, \ldots, \dim(G)$ is a fixed chart about the identity $e \in G$. Then $(\phi_\alpha(U_\alpha \times V), \bar{x}_i, \bar{y}^a)$ is a chart on P with $\bar{x}_i(u) = x_i \circ \pi(u)$, $\bar{y}^a(u) = y^a(\phi^{-1}_{\alpha\pi(u)}(u))$. There are local equivariant vector fields E_i and E^a on $P|\,U_\alpha$ defined by

$$E_i(u) = dR_{\phi^{-1}_{\alpha x}(u)}|_{u_0}\left(\frac{\partial}{\partial \bar{x}_i}\,|_{u_0}\right)$$

$$E^a(u) = dR_{\phi^{-1}_{\alpha x}(u)}|_{u_0}\left(\frac{\partial}{\partial \bar{y}^a}|_{u_0}\right)$$

where $x = \pi(u)$ and $u_0 = \phi_{\alpha x}(e)$. Note that if Z is an equivariant vector field on P, then its local expression $Z = Z_i E_i + Z^a E^a$ on $P \mid U_\alpha$ has component functions Z_i, Z^a which only depend on x : $Z_i = \bar{Z}_i \circ \pi$ where $Z_i(x) = \bar{Z}_i(\phi_{\alpha x}(e))$ (similarly for Z^a). In the sequel we will identify functions $\bar{F}$ on M with their pullbacks $F = \bar{F} \circ \pi$ to P (i.e. $F = \bar{F}$).

If h is an Ehresmann connection then

$$h(E^a) = 0$$

$$h(E_i) = E_i - A_i^a(h)E^a$$

where

$$A_i^a(h)(x) = -h_{u_0}\left(\frac{\partial}{\partial \bar{x}_i}|_{u_0}\right)(\bar{y}^a)$$

are the local components for h (the minus sign is chosen for convenience in the ensuing formulas).

The corresponding local expression for a connection $\sigma : M \to C$ is as follows. Let $h = \mu^{-1}(\sigma)$, $A_i^a(\sigma) = A_i^a(h)$. For a vector field X on M, the section $\sigma(X) : M \to TP/G$, $(\sigma(X)(x) = \sigma_x(X)_x))$ corresponds to an equivariant vector field $X^\sigma : P \to TP$ on P : $X^\sigma = \mu^{-1}(\sigma(X))$ is called the *horizontal lift* of X $(h(X^\sigma) = X^\sigma$, $d\pi|_u X^\sigma = X_{\pi(u)})$. Locally one has

$$X^\sigma = X_i(E_i - A_i^a(\sigma)E^a),$$

$X_i = X(x_i)$, and letting $e_i = \mu(E_i)$, $e^a = \mu(E^a)$ be the corresponding local sections of TP/G one finds that

$$\sigma(X) = X_i(e_i - A_i^a(\sigma)e^a)$$

locally on U_α. As a 1-form on M with values in $\Gamma(TP/G)$ the local expression for σ is

$$\sigma = (e_i - A_i^a(\sigma)\, e^a) \otimes dx_i$$

Using any of the various means for computing the curvature (say $F^\sigma(X, Y) = \mu([X, Y]^\sigma - [X^\sigma, Y^\sigma])$) one finds that locally

$$F^\sigma = F_{ij}^c(\sigma)\, e^c \otimes dx_i\, dx_j$$

where

$$F_{ij}^c(\sigma) = \partial A_j^c/\partial x_i - \partial A_i^c/\partial x_j - m_{ab}^c A_j^a A_i^b$$

Here the m^c_{ab} are the structure constants of G relative to the basis T^a where $(T^a)_g \equiv dL_g|_e(\partial/\partial y^a|_e)$.

The component expression for $\nabla^\sigma \tau$, $\tau \in \Gamma(E)$ is as follows. Let $\psi_\alpha : U_\alpha \times F \to E \mid U_\alpha$ be the local trivialization given by $\psi_\alpha(x, \xi) = [\phi_{\alpha x}(e), \xi]$, and let $\{\epsilon_k\}$ be a basis for F with coefficient functionals $\{\epsilon^*_k\}$ (basis for F^*). Then $\tau = \tau^k \bar{\epsilon}_k$ where

$$\tau^k(x) = \epsilon^*_k(\psi^{-1}_{\alpha x}(\tau(x)))$$

and

$$\bar{\epsilon}_k(x) = \psi_{\alpha x}(\epsilon_k)$$

The action $B : G \times F \to F$ gives a matrix representation of T^a on Hom(F), again denoted by T^a, with entries

$$T^a_{km} = \partial B_{km}(e)/\partial y^a$$

where $B_{km}(g) \equiv \epsilon^*_k(B(g, \epsilon_m))$. Then a short computation, using $\nabla^\sigma \tau(X) = \mu(D^h\mu^{-1}(\tau)(X^\sigma))$, gives

$$\nabla^\sigma \tau = [\partial\tau^k/\partial x_i + A^a_i(\sigma)\, T^a_{km}\tau^m]\,\bar{\epsilon}_k \otimes dx_i \tag{5.1}$$

VI. COMMENTS ON THE MAP B_γ AND THE UTIYAMA EXTENSION PROCESS

One sees from the coordinate expression (5.1) that the map $B_\gamma([\tau]_x) = (\tau(x), \nabla^\gamma\tau(x))$ is well-defined and is injective. To see that B_γ is surjective, suppose $(z, z') \in E \oplus A^1(M, E)$ and let $x \equiv \pi_E(z)$, $z^k \equiv \epsilon^*_k(\psi^{-1}_{\alpha x}(z))$, $z^k_i \equiv \epsilon^*_k[\psi^{-1}_{\alpha x}(z'(\partial/\partial x_i|_x))]$ (z^k, z^k_i are the local coordinates of z, z'). Then define a local section $\bar{\tau} : U_\alpha \to E$ by

$$\bar{\tau}(\bar{x}) = \psi_{\alpha\bar{x}}(z^k\epsilon_k + [z^k_i - A^a_i(\gamma)(x)T^a_{km}\, z^m](x_i(\bar{x}) - x_i(x))\epsilon_k) \tag{6.1}$$

then

$$\bar{\tau}^k(x) = z^k$$

$$\frac{\partial\bar{\tau}^k}{\partial x_i}(x) = z^k_i - A^a_i(\gamma)(x)T^a_{km}z^m$$

and consequently $B_\gamma([\bar{\tau}]_x) = (z, z')$. This shows that B_γ is surjective and that

$$B^{-1}_\gamma(z, z') = [\bar{\tau}]_x$$

where $\bar{\tau}$ is defined by (6.1). From this we get as well a coordinate expression for the Utiyama map:

$$U_\gamma([\tau]_x, [\sigma]_x) = B_\gamma^{-1}\ (\tau(x), \nabla^\sigma \tau(x))$$
$$\equiv [\bar{\tau}]_x$$

where $\bar{\tau}$ is a local section of E with

(6.2) $$\bar{\tau}(x) = \tau(x)$$
$$\frac{\partial \bar{\tau}^k}{\partial x_i}(x) = \frac{\partial \tau^k}{\partial x_j}(x) + [A_i^a(\sigma)(x) - A_i^a(\gamma)(x)]T^a_{km}\,\tau^m(x)$$

Formula (6.2) indicates the extent to which the extension $L_1^\gamma = L_1 \circ U_\gamma$ is an appropriate generalization of Utiyama's extension (Yang-Mills trick): for trivial bundles $P = M \times G$, the extension by the trivial connection γ ($A_i^a(\gamma) = 0$) has roughly the effect of obtaining L_1^γ from L_1 by replacing the ordinary derivative of τ by covariant derivatives.

The invariance of L_1 under the set

$$G_\gamma = \{B_\gamma^{-1} \circ \rho \circ B_\gamma \mid \rho \in GA(P)_{\text{loc}}\}$$

of bundle maps of J^1E is clarified by the following calculations. Suppose $\rho : P\,|\,W \to P\,|\,W$ is a local gauge transformation (so that $\rho(u) = u\lambda(u)$, $\lambda : P|\,W \to G$, $\lambda(ug) = g^{-1}\lambda(u)g$), then on $U_\alpha \subset W$, λ is given by $\lambda(u) = \phi_{\alpha x}^{-1}(u)^{-1}\lambda_\alpha(x)\,\phi_{\alpha x}^{-1}(u)$ where $x = \pi(u)$ and $\lambda_\alpha(x) = \lambda(\phi_{\alpha x}(e))$. A short calculation shows that for $(z, z') \in E^1\,|\,U_\alpha$, $\rho(z, z') = (\rho(z), \rho(z'))$ has coordinate expression

$$\rho(z)^k = B_{km}(\lambda_\alpha(x))z^m$$
$$\rho(z')_i^k = B_{km}(\lambda_\alpha(x))\, z_i^m$$

($x = \pi_E(z)$). Using that, together with the foregoing, one finds that

$$[\bar{\tau}]_x \equiv B_\gamma^{-1} \circ \rho \circ B_\gamma([\tau]_x)$$

is represented by a local section $\tau : U_\alpha \to E$ with

$$\bar{\tau}^k(x) = B_{km}(\lambda_\alpha(x))\tau^m(x)$$

and

$$\frac{\partial \bar{\tau}^k}{\partial x_i}(x) = B_{km}(\lambda_\alpha(x))\tau^m(x) + A_i^a(\gamma)(x)[B(\lambda_\alpha(x)), T^a]_{km}\,\tau^m(x)$$

where $B(\lambda_\alpha(x))$ is the matrix with entries $B_{km}(\lambda_\alpha(x))$ and [. , .] is the usual commutator bracket. Thus in Utiyama's case ($P = M \times G$, γ trivial, $A_i^a(\gamma) = 0$) its not hard to see that invariance under G_γ is equivalent to global gauge invariance).

VII. PROOF OF LEMMA 1

The final detail that remains is the proof of Lemma 1 (Section II). For this first define a local bundle map $\Lambda_\alpha : A^2(M, AdP)|\,U_\alpha \to J^1 C$ as follows. A w in the former bundle (in the fiber over $x \in U_\alpha$) has coordinates: $w^a_{ij} = \tilde{y}^a[w(\partial/\partial x_i|_x, \partial/\partial x_j|_x)]$ where $\tilde{y}^a$ is the fiber coordinate function on AdP. Define $f^a_i : U_\alpha \to \mathbb{R}$ by $f^a_i(\bar{x}) = -w^{*a}_{ij}[x_j(\bar{x}) - x_j(x)]$ where $w^{*a}_{ij} = w^a_{ij}$ if $i < j$ and $w^{*a}_{ij} = 0$ if $i \geqslant j$. Then

$$\bar{\sigma} = [e_i - f^a_i e^a] \quad dx_j$$

defines a local section $\bar{\sigma} : U_\alpha \to C$ with

$$(7.1) \qquad \begin{aligned} &A^a_i(\bar{\sigma})(x) = 0 \\ &\partial A^a_i(\bar{\sigma})(x)/\partial x_j = -w^{*a}_{ij} \end{aligned}$$

Consequently $F^{\bar{\sigma}}(x) = w$ and so we take

$$\Lambda_\alpha(w) \equiv [\bar{\sigma}]_x$$

From the construction it then follows that $\Omega \circ \Lambda_\alpha = 1$, so Ω is an epimorphism (Cf. Garcia [3] for the same construction). Now extend Garcia's map Λ_α to $E^1 \oplus A^2(M, AdP)|\,U_\alpha$ by

$$\Lambda^+_\alpha(z, z', w) \equiv (B^{-1}_{\Lambda_\alpha(w)}(z, z'), \Lambda_\alpha(w))$$

Then its easy to see that

$$\Omega^+ \circ \Lambda^+_\alpha = 1$$

(so Ω^+ is an epimorphism). Next suppose $([\tau]_x, [\sigma]_x) \in J^1(E \oplus C)|\,U_\alpha$ and let

$$([\bar{\tau}]_x, [\bar{\sigma}]_x) = \Lambda^+_\alpha \circ \Omega^+([\tau]_x, [\sigma]_x)$$

Then by the foregoing (Cf. Eqs. (6.2) and (7.1))

$$A^a_i(\bar{\sigma})(x) = 0$$

$$\partial A^a_i(\bar{\sigma})(x)/\partial x_i = \begin{Bmatrix} F^a_{ji}(\sigma)(x) & i < j \\ 0 & i \geqslant j \end{Bmatrix}$$

and

$$\bar{\tau}(x) = \tau(x)$$

$$\frac{\partial \bar{\tau}^k}{\partial x_i}(x) = \frac{\partial \tau^k}{\partial x_i}(x) + A^a_i(\sigma)(x)\, T^a_{km}\, \tau^m(x)$$

Now since $\rho^1([\sigma]_x, [\tau]_x) = ([\rho^{-1*}\sigma]_x, [\rho^{-1*}\tau]_x)$ for any local gauge transformation ρ, the completion of the proof follows from:

LEMMA 2. Suppose σ is a connection and $x \in M$. Then relative to a chart U_α about x there exists a local gauge transformation ρ such that

$$A_i^a(\rho^{-1*}\sigma)(x) = 0$$

$$\partial A_i^a(\rho^{-1*}\sigma)(x)/\partial x_j = \begin{cases} F_{ji}^a(\sigma)(x) & i<j \\ 0 & i \geqslant j \end{cases}$$

Furthermore for any section τ of E

$$(\rho^{-1*}\tau)(x) = \tau(x)$$

$$\partial(\rho^{-1*}\tau)^k(x)/\partial x_i = \frac{\partial\tau^k}{\partial x_i}(x) + A_i^a(\sigma)(x)\, T_{km}^a\, \tau^m(x)$$

Proof: Let $K_i^a = A_i^a(\sigma)(x)$, $K_{ij}^a = \partial A_i^a(\sigma)(x)/\partial x_j$

$$H_{ij}^a = \begin{bmatrix} 1/2[K_{ji}^a + C_{db}^a K_i^d K_j^b] \text{ if } i<j \\ 1/2[K_{ij}^a + C_{db}^a K_j^d K_i^b] \text{ if } i \geqslant j \end{bmatrix}$$

where $C_{db}^a \equiv \partial^2 B^a(e, e)/\partial y_1^d\, \partial y_2^b$; $B : G \times G \to G$ is the group multiplication (so the structure constants $m_{db}^a = C_{db}^a - C_{bd}^a$). On a suitably smaller neighborhood $U'_\alpha \subset U_\alpha$ about x we can define a map $\lambda_\alpha : U'_\alpha \to G$ such that $\lambda_\alpha^a \equiv y^a \circ \lambda_\alpha$ is given by

$$\lambda_\alpha^a(\bar{x}) = \begin{bmatrix} y^a(e) + K_i^a[x_i(\bar{x}) - x_i(x)] \\ \qquad + H_{ij}^a[x_i(\bar{x}) - x_i(x)][x_j(\bar{x}) - x_j(x)] \end{bmatrix}$$

Then define $\lambda : P|\,U'_\alpha \to G$ by

$$\lambda(\bar{u}) = \phi_{\alpha\bar{x}}^{-1}(\bar{u})^{-1}\, \lambda_{\alpha\bar{x}}(\bar{x})\, \phi_{\alpha\bar{x}}^{-1}(\bar{u})$$

where $\bar{x} = \pi(\bar{u})$. This gives a local gauge transformation $\rho : P|\,U'_\alpha \to P|\,U'_\alpha : \rho(\bar{u}) \equiv \bar{u}\lambda(\bar{u})$. If $h = \mu^{-1}(\sigma)$ is the corresponding Ehresmann connection, then by examining the component expressions $A_i^a(\rho^{-1*}h) = A_i^a(\rho^{-1*}\sigma)$ for $\rho^{-1*}h$ one finds (for any gauge transformation ρ) that

$$A_i^a(\rho^{-1*}\sigma) = -M_i^a + N_b^a A_i^b(\sigma)$$

where

$$M_i^a(\bar{x}) = \frac{\partial B^a}{\partial y_1^c}(\lambda_\alpha(\bar{x}), \lambda_\alpha(\bar{x})^{-1})\, \frac{\partial \lambda_a^c}{\partial x_i}(\bar{x})$$

$$N^a_b(\bar{x}) = \frac{\partial B^a}{\partial y^c_1}(\lambda_\alpha(\bar{x}), \lambda_\alpha(\bar{x})^{-1})\ \frac{\partial B^c}{\partial y^b_2}(\lambda_\alpha(\bar{x}), e)$$

This rest of the proof is now a straight-forward calculation from this. ∎

VIII. CONCLUSION

In conclusion, we outline the vector field approach to Theorem 3, which is along the lines of Garcia's proof of Utiyama's theorem (Theorem 5.1 in Ref. 3). In addition, we briefly describe now the approach due to Mangiarotti and Modugno, using adjoint forms and jet shift differentials, encompasses all of this as well, [18, 19, 20].

The representation of the automorphism group of P as bundle maps on associated bundles carries over to vector fields as well. In particular, suppose that Z is an infinitesimal gauge transformation (a vertical right invariant vector field on P) with flow ρ_t, and let $\bar{\rho}_t$ be the representation of ρ_t on the interaction bundle $R \equiv E \oplus C$, with $\bar{Z}(r) = [d\bar{\rho}_t(r)/dt]\,|_{t=0}$ the corresponding infinitesimal generator. This representation $\bar{Z}$ of Z on R prolongs, in the usual way, to a representation $\bar{Z}^1$ of Z as a vertical vector field on J^1R. Then it is easy to see that a Lagrangian $L^+ : J^1R \to \mathbb{R}$ is locally gauge invariant if and only if $\bar{Z}^1(L^+) = 0$ for every infinitesimal gauge transformation Z. In terms of a standard chart on J^1R (with coordinates x_i, z^k, A^a_i, z^k_j, A^a_{ij}) the component expression for $\bar{Z}^1$ works out to be

$$\bar{Z}^1 = W^a U^a + (\partial W^a/\partial x_i)U^a_i + \Sigma_{i<j}(\partial^2 W^a/\partial x_i \partial x_j)U^a_{ij}$$

where $W^a(x) = Z(\bar{y}^a)(\phi_{\alpha x}(e))$ and

$$U^a_{ij} = \partial/\partial A^a_{ij} + \partial/\partial A^a_{ji}$$

$$U^a_i = -\,\partial/\partial A^a_i + (T^a_{kp}z^p)\,\partial/\partial z^k_i + (m^c_{ab}A^b_j)\,\partial/\partial A^c_{ji}$$

$$U^a = T^a_{kp}(z^p\ \partial/\partial z^k + z^p_i\ \partial/\partial z^k_i) + m^c_{ab}(A^b_i\,\partial/\partial A^c_i + A^b_{ij}\partial/\partial A^c_{ij})$$

Now with a little work one can show that local gauge invariance of L^+ is equivalent to the three local conditions: (C_0) $U^a_{ij}(L^+) = 0$, (C_1) $U^a_i(L^+) = 0$, and (C_2) $U^a(L^+) = 0$. Further work shows that (C_0) and (C_1) imply that L^+ factors: $L^+ = K^+ \circ \Omega^+$, and then (C_2) gives that K^+ is gauge invariant (and conversely). Thus (modulo a good many details) one gets a proof of Theorem 3. This is Garcia's approach [3] in proving Utiyama's theorem.

The contribution of Mangiarotti and Modugno to the theory was the recognition that the local conditions (C_0), (C_1), (C_2) arise from global conditions on certain adjoint forms and jet shift differentials connected with L^+. We briefly sketch the ideas behind their work.

Let $\pi_k : J^k R \to M$ be the k-th order jet bundle for $R \equiv E \oplus C$, and for a vector bundle Q over M, let $A^r_H(J^k R, Q) \simeq \Lambda^r T^*M \otimes Q$ denote the bundle over $J^k R$ with elements (z, θ_z), where $z \in J^k R$ and $\theta_z : T_z J^k R \times \ldots \times T_z J^k R \to Q_x$, $(x = \pi_k(z))$ is an r-linear form which vanishes whenever the of its arguments is vertical (in $\ker d\pi_k|_z$).

The sections $Q^{(k,r)} \equiv \Lambda^r_H(J^k R, Q)$ of this bundle are the horizontal, Q-valued, r-forms on $J^k R$. In particular $AdP^{(k,r)}$ and $Ad^*P^{(k,r)}$ are the adjoint and co-adjoint r-forms. Now, assuming that Q is an associated bundle, each section $s = (\tau, \sigma) : M \to R$ gives rise to a diagram

$$Q^{(k,r)} \to \Lambda^r(M, Q) \to \Lambda^{r+1}(M, Q) \to Q^{(k+1,r+1)}$$

where the mapping (arrows) are $j_k(s)^*$, ∇^σ, and π^*_{k+1} respectively. This gives the jet shift differential $d : Q^{(k,r)} \to Q^{(k+1,r+1)}$ defined by

$$d\theta_z = \pi^*_{k+1} \circ \nabla^\sigma \circ j_k(s)^*(\theta)\big|_z$$

where $z = [s]_x = j_{k+1}(s)(x)$. An additional key element in the theory is the mapping $\beta^* : V^*Q^{(k,r)} \to Ad^*P^{(k,r)}$ which arises from the representation of infinitesimal gauge transformations as vertical vector fields on associated bundles.

Now suppose $\lambda \in \Lambda^m_H(J^1 R)$ is a Lagrangian form ($m = \dim M$) (when M has a volume form Δ then λ globally has the form $\lambda = L^+ \ \pi^*_1(\Delta)$; otherwise λ only has this form locally). Due to the product nature of $R = E \oplus C$, one gets decompositions $m = (m_E, m_C)$ and $e = (e_E, e_C)$ of the momentum form and Euler-Lagrange form for λ, with

$$m_E \in V^*E^{(1,m-1)}, \quad m_C \in V^*C^{(1,m-1)} \supset Ad^*P^{(1,m-2)}$$

$$e_E \in V^*E^{(2,m)}, \quad e_C \in V^*C^{(2,m)} \simeq Ad^*P^{(2,m-1)}.$$

Based on these constructions Mangiarotti and Modugno recognized that the local gauge invariance of λ is equivalent to the following three conditions (Cf. Ref's 18, 19, 20):

(C_0) $\quad m_C \in Ad^*P^{(1,m-2)}$

(C_1) $\quad e_C = dm_C - \beta^*(m_E)$

(C_2) $\quad de_C = \beta^*(e_E)$.

These conditions generalize nicely their previous conditions (Ref. 9) for free gauge field Lagrangians (since one can take $E = 0$, so that $m_E = 0 = e_E$ in the above). Their work also contains other results, such as the factorization of λ through the extended curvature map and the minimal interaction condition.

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